

CLIFFORD ALGEBRAS AND SPINORS FOR ARBITRARY BRAIDS

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ABSTRACT. General braided counterparts of classical Clifford algebras are introduced and investigated. Braided Clifford algebras are defined as Chevalley-Kähler deformations of the corresponding braided exterior algebras. Analogs of the spinor representations are studied, generalizing classical Cartan's approach. It is shown that, under certain assumptions concerning the braiding, the spinor representation is faithful and irreducible, as in the classical theory.

1. INTRODUCTION

In this study classical theory of Clifford algebras and spinors will be incorporated into a general braided framework. The main idea that will be followed is the Chevalley-Kähler interpretation of Clifford algebras, as deformations of exterior algebras. General braided Clifford algebras will be introduced by constructing a new product in the braided exterior algebras spaces.

The paper is organized as follows. In the next section the main properties of braided exterior algebras [6] are collected. In Section 3 we introduce and analyze a braided inner product. In particular, we deal with both exterior algebras over the space and its dual. These algebras are mutually dual in a natural manner. It turns out that the inner product with elements of the dual exterior algebra can be viewed as the transposed operator of the corresponding exterior multiplication, as in the classical theory.

Section 4 is devoted to the construction and general analysis of braided Clifford algebras. Our construction conceptually follows classical Chevalley-Kähler approach. We introduce a new product in the exterior algebra space. This product is expressible in terms of the exterior product, and various "relative" contractions, which are constructed from the corresponding scalar product in the vector space and the braided inner product. In such a way the Clifford algebra becomes a deformation of the exterior algebra. We also introduce an analog of the Crumeyrolle map [3] in the tensor algebra, connecting Clifford and exterior ideals. This can be viewed as another way to define braided Clifford algebras.

In Section 5 we study counterparts of algebraic spinors, conceptually following Cartan's geometrical approach, and varying and generalizing a construction given in [1]. Spinors are defined as elements of braided exterior algebras over certain isotropic subspaces of the initial vector space. By construction the spinor space is a left Clifford module. We prove that, under certain assumptions concerning the

braid, the spinor representation is irreducible and faithful, as in the classical theory. Finally, in Section 6 some concluding remarks are made.

2. ON BRAIDED EXTERIOR ALGEBRAS

In this section the main properties of exterior algebras [6] associated to an arbitrary braid operator are collected. Let W be a (complex) finite-dimensional vector space, and let $\sigma: W \otimes W \rightarrow W \otimes W$ be a bijective map satisfying the braid equation

$$(1) \quad (\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma).$$

Let $A: W^{\otimes} \rightarrow W^{\otimes}$ be the corresponding total antisymmetrizer map. Its components $A_n: W^{\otimes n} \rightarrow W^{\otimes n}$ are given by

$$A_n = \sum_{\pi \in S_n} (-1)^\pi \sigma_\pi$$

where $\sigma_\pi: W^{\otimes n} \rightarrow W^{\otimes n}$ are maps obtained by replacing transpositions figuring in a minimal decomposition of π by the corresponding σ -twists. The following identities hold

$$(2) \quad A_{n+k} = (A_n \otimes A_k) A_{nk}$$

$$(3) \quad A_{n+k} = B_{nk} (A_n \otimes A_k),$$

where

$$A_{nk} = \sum_{\pi \in S_{nk}} (-1)^\pi \sigma_{\pi^{-1}}, \quad B_{nk} = \sum_{\pi \in S_{nk}} (-1)^\pi \sigma_\pi$$

and $S_{nk} \subseteq S_{n+k}$ is the set of permutations preserving the order of sets $\{1, \dots, n\}$ and $\{n+1, \dots, n+k\}$.

By definition [6], the corresponding braided *exterior algebra* $W^\wedge(\sigma) = W^\wedge$ is the factoralgebra of the tensor algebra W^{\otimes} relative to the ideal $\ker(A)$.

The algebra W^\wedge can be naturally realized as a subspace $\text{im}(A)$ in W^{\otimes} . This realization is given by

$$(4) \quad [\psi + \ker(A)] \leftrightarrow A(\psi).$$

In terms of the above identification the exterior product of $\psi \in W^{\wedge n}$ and $\varphi \in W^{\wedge k}$ is given by

$$(5) \quad \psi \wedge \varphi = B_{nk}(\psi \otimes \varphi).$$

3. BRAIDED INNER PRODUCT

The spaces $W^{*\otimes n}$ and $W^{\otimes n}$ are mutually dual, in a natural manner. The duality between them is given by

$$(6) \quad \langle f_1 \otimes \dots \otimes f_n, \psi_n \otimes \dots \otimes \psi_1 \rangle = f_1(\psi_1) \dots f_n(\psi_n).$$

This pairing trivially extends to the whole tensor algebras. We will assume that the dual space is equipped with the transposed braiding $\sigma^*: W^* \otimes W^* \rightarrow W^* \otimes W^*$. In what follows operators associated to σ^* will be endowed with the star.

Maps $\{A_n, A_n^*\}$, as well as $\{A_{nk}, B_{kn}^*\}$ and $\{A_{nk}^*, B_{kn}\}$ are mutually transposed. Furthermore, it is possible to define a natural pairing $(\)^\wedge$ between exterior algebras $W^{*\wedge}$ and W^\wedge . Explicitly,

$$(\phi, \vartheta)^\wedge = \langle \phi, \tilde{\vartheta} \rangle,$$

where $W^{*\wedge}$ is embedded in $W^{*\otimes}$, and $[\tilde{\vartheta}]^\wedge = \vartheta$.

For each $f \in W^*$ and $\xi \in W^{\wedge n}$, let $f \sqcup \xi \in W^{\wedge n-1}$ be an element given by

$$(7) \quad f \sqcup \xi = (f \otimes \text{id}^{n-1})(\xi).$$

In the above formula, it is assumed that W^\wedge is embedded in W^\otimes . The fact that $f \sqcup \xi$ belongs to W^\wedge follows from the decomposition (2).

In such a way we have constructed a map $\sqcup: W^* \otimes W^\wedge \rightarrow W^\wedge$ (a counterpart of the standard inner product with 1-forms). For each $f \in W^*$ we will denote by $\sqcup_f: W^\wedge \rightarrow W^\wedge$ the corresponding contraction map.

In what follows it will be assumed that σ is naturally extended to a braiding on $W^\wedge \otimes W^\wedge$, by requiring

$$(8) \quad \sigma(m \otimes \text{id}) = (\text{id} \otimes m)(\sigma \otimes \text{id})(\text{id} \otimes \sigma)$$

$$(9) \quad \sigma(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}),$$

where $m: W^\wedge \otimes W^\wedge \rightarrow W^\wedge$ is the product map.

Lemma 1. *The following identity holds*

$$\sqcup_f(\xi\eta) = \sqcup_f(\xi)\eta + (-1)^{\partial\xi} m\sigma^{-1}(\sqcup_f \otimes \text{id})\sigma(\xi \otimes \eta).$$

This is a counterpart of the standard graded Leibniz rule.

Proof. The statement follows from the definition of \sqcup , decomposition (2), and correspondence (4). \square

The introduced operator \sqcup can be trivially extended to the map of the form $\sqcup: W^{*\otimes} \otimes W^\wedge \rightarrow W^\wedge$ such that

$$\sqcup_{f \otimes g} = \sqcup_f \sqcup_g$$

for each $f, g \in W^{*\otimes}$.

Lemma 2. *If $f \in \ker(A^*)$ then $\sqcup_f = 0$.*

Proof. The statement follows from decomposition (2), and the definition of \sqcup . \square

Hence, we can pass from $W^{*\otimes}$ to $W^{*\wedge}$ in the first argument of \sqcup . In such a way we obtain a map of the form $\sqcup: W^{*\wedge} \otimes W^\wedge \rightarrow W^\wedge$ (we use the same symbol for different maps, because the domain is clear from the context).

Definition 1. The constructed map is called *the braided inner product*.

The inner product map can be equivalently described as the transposed exterior multiplication. For each $f \in W^{*\wedge}$, let $\wedge_f: W^{*\wedge} \rightarrow W^{*\wedge}$ be a linear map given by $\wedge_f(\phi) = \phi \wedge f$.

Proposition 3. *The following identity holds*

$$(\phi, \sqcup_f(\xi))^\wedge = (\wedge_f(\phi), \xi)^\wedge.$$

In other words, \sqcup_f and \wedge_f are mutually transposed.

Proof. If $\phi \in W^{*\wedge k}$ and $\xi \in W^{\wedge k+r}$ then

$$\begin{aligned} (\wedge_f(\phi), \xi)^\wedge &= (\phi \wedge f, \xi)^\wedge = \langle B_{kr}^*(\phi \otimes f), \tilde{\xi} \rangle \\ &= \langle \phi \otimes f, A_{rk}\tilde{\xi} \rangle = (\phi, f \sqcup \xi)^\wedge = (\phi, \sqcup_f(\xi))^\wedge \end{aligned}$$

for each $f \in W^{*\wedge r}$. \square

4. BRAIDED CLIFFORD ALGEBRAS

Let us assume that W is endowed with a scalar product F , understood as a linear map $F: W \otimes W \rightarrow \mathbb{C}$. Let $\ell_F: W \rightarrow W^*$ be an associated correlation, given by

$$[\ell_F(x)](y) = F(x \otimes y).$$

As first, we are going to introduce various types of contraction operators in W^\wedge which will play a fundamental role in constructing the corresponding Clifford algebra.

In what follows it will be assumed that F and σ are mutually related such that the following “funny functoriality” holds

$$(10) \quad (F \otimes \text{id})(\text{id} \otimes \sigma) = (\text{id} \otimes F)(\sigma \otimes \text{id}).$$

This implies

$$(11) \quad (\ell_F \otimes \ell_F)\sigma = \sigma^*(\ell_F \otimes \ell_F).$$

In particular

$$(12) \quad \ell_F^\otimes(\ker(A)) \subseteq \ker(A^*),$$

where ℓ_F^\otimes is the unital multiplicative extension of ℓ_F . Factorizing ℓ_F^\otimes through ideals $\ker\{A, A^*\}$ we obtain a homomorphism $\ell_F^\wedge: W^\wedge \rightarrow W^{*\wedge}$.

Let $\iota^F: W^\wedge \otimes W^\wedge \rightarrow W^\wedge$ be a contraction map given by

$$(13) \quad \iota^F = \sqcup(\ell_F^\wedge \otimes \text{id}).$$

By construction, ι^F is multiplicative on the first factor and satisfies the following braided variant of the Leibniz rule

$$\iota_x^F(\vartheta\eta) = \iota_x^F(\vartheta)\eta + (-1)^{\partial\vartheta} \sum_k \vartheta_k \iota_{x_k}^F(\eta)$$

where $x \in W$ and $\sum_k \vartheta_k \otimes x_k = \sigma(x \otimes \vartheta)$.

Let us define relative contraction operators $\langle \rangle_k: W^\wedge \times W^\wedge \rightarrow W^\wedge$ as follows

$$\langle \zeta, \xi \rangle_k = \sum_j \psi_j \wedge (\iota_{\varphi_j}^F \xi)$$

Here, it is assumed that $\zeta \in W^{\wedge n}$ and $\sum_j \psi_j \wedge \varphi_j = [A_{n-kk} \tilde{\zeta}]^\wedge$, where $\varphi_j \in W^{\wedge k}$, $\psi_j \in W^{\wedge n-k}$ and $\tilde{\zeta} \in W^{\otimes n}$ satisfies $[\tilde{\zeta}]^\wedge = \zeta$. Consistency of this definition follows from (2). If $n < k$ we define $\langle \rangle_k = 0$.

Now, we define a *new product* on W^\wedge , in the spirit of the classical construction. This product is introduced by the following expression

$$\psi \circ \varphi = \psi \wedge \varphi + \sum_{k \geq 1} \langle \psi, \varphi \rangle_k.$$

In particular,

$$x \circ \psi = x \wedge \psi + \iota_x^F(\psi)$$

for $x \in W$. This generalizes classical Chevalley's formula.

Theorem 4. *Endowed with \circ , the space W^\wedge becomes a unital associative algebra, with the unity $1 \in W^\wedge$.*

Proof. The standard diagrammatic computations, using braid diagrams and functoriality property (10). \square

Definition 2. The algebra $\mathfrak{cl}(W, \sigma, F) = (W^\wedge, \circ)$ is called *the braided Clifford algebra* (associated to $\{W, \sigma, F\}$).

The constructed algebra can be understood as a deformation of the exterior algebra W^\wedge . The graded algebra associated to the filtered algebra $\mathfrak{cl}(W, \sigma, F)$ naturally coincides with W^\wedge .

The algebra $\mathfrak{cl}(W, \sigma, F)$ can be viewed as a factoralgebra $\mathfrak{cl}(W, \sigma, F) = W^\otimes / J_F$, where J_F is the kernel of the canonical epimorphism $j_F: W^\otimes \rightarrow \mathfrak{cl}(W, \sigma, F)$ (extending the identity map on W). Now, this ideal will be described in an independent way, using a generalization of the construction presented by Crumeyrolle in [3].

As first, a Clifford product in the tensor algebra W^\otimes will be introduced. Let us consider a linear map $\lambda_F: W^\otimes \rightarrow W^\otimes$ defined by

$$\lambda_F(1) = 1 \quad \lambda_F(x \otimes \vartheta) = x \otimes \lambda_F(\vartheta) + \iota_x^F \lambda_F(\vartheta)$$

where $x \in W$ and $\vartheta \in W^\otimes$. In the above formula, ι_x^F is considered as a braided antiderivation on W^\otimes . The map λ_F is bijective. Let \circ be a new product in W^\otimes , given by

$$\vartheta \circ \eta = \lambda_F(\lambda_F^{-1}(\vartheta) \otimes \lambda_F^{-1}(\eta)).$$

By construction the space $\ker(A)$ is a left ideal in W^\otimes , relative to this new product. Condition (10) ensures that $\ker(A)$ is also a right \circ -ideal.

Theorem 5. *We have*

$$(W^\otimes, \circ) / \ker(A) = \mathfrak{cl}(W, \sigma, F).$$

Proof. It is sufficient to observe that (10) implies that the product \circ on W^\otimes is given by essentially the same formula as for W^\wedge , the only difference is that \wedge should be replaced by \otimes , and contractions are acting on W^\otimes . \square

In other words, the factorization map $[\]^\wedge: W^\otimes \rightarrow W^\wedge$ is also a homomorphism of corresponding deformed algebras. The map λ_F is a braided counterpart of the maps introduced in [3] and [5].

Lemma 6. *We have*

$$(14) \quad \ker(A) = \lambda_F(J_F).$$

Proof. The statement follows from the equality $[\]^\wedge \lambda_F = j_F$. \square

5. THE SPINOR REPRESENTATION

This section is devoted to a braided generalization of classical Cartan theory of spinors [2]. We follow ideas of [1]. Let us assume that the space W is splitted into a direct sum

$$W = W_1 \oplus W_2$$

where W_1, W_2 are F -isotropic subspaces. Furthermore, let us assume that this decomposition is compatible with the braiding σ in the following way

$$(15) \quad \sigma(W_i \otimes W_j) = W_j \otimes W_i$$

$$(16) \quad \sigma^2 \Big| \left\{ (W_1 \otimes W_2) \oplus (W_2 \otimes W_1) \right\} = \text{id}.$$

Finally, it will be assumed that $F \upharpoonright (W_1 \otimes W_2) = 0$ and that $F \upharpoonright (W_2 \otimes W_1)$ is *nondegenerate*. In this case, $W_2 = W_1^*$, in a natural manner. The duality is given by $F(f \otimes x) = f(x)$, for $f \in W_2$ and $x \in W_1$.

Exterior algebras W_1^\wedge and W_2^\wedge are understandable as subalgebras of $\mathfrak{cl}(W, \sigma, F)$, in a natural manner.

Lemma 7. *The map $\mu: W_1^\wedge \otimes W_2^\wedge \rightarrow \mathfrak{cl}(W, \sigma, F)$ defined by*

$$(17) \quad \mu(u \otimes v) = u \circ v$$

is bijective.

Proof. If $u \in W_1$ and $v \in W_2$ then

$$vu + \sum_k u_k v_k - F(v \otimes u)1 = 0$$

where $\sum_k u_k \otimes v_k = \sigma(v \otimes u)$. This implies that μ is surjective.

Let $\mu^*: W_1^\wedge \otimes W_2^\wedge \rightarrow W^\wedge$ be the grade-preserving component of μ . This map is explicitly given by $\mu^*(u \otimes v) = u \wedge v$. We prove that μ^* is injective. We have

$$\mu^*(u \otimes v) = B_{kl}(u \otimes v)$$

for $u \in W_1^{\wedge k}$ and $v \in W_2^{\wedge l}$. If $\psi \in W_1^{\wedge k} \otimes W_2^{\wedge l}$ then $p_{kl}\mu^*(\psi) = \psi$, where $p_{kl}: W^{\otimes k+l} \rightarrow W_1^{\otimes k} \otimes W_2^{\otimes l}$ is the projection map. Hence μ^* is injective. \square

Let us consider the space $\mathcal{K} = W_2^\wedge$, and let $\kappa: \mathcal{K} \rightarrow \mathbb{C}$ be a natural character, specified by $\kappa(1) = 1$ and $\kappa(W_2) = \{0\}$. This gives a left \mathcal{K} -module structure on the number field \mathbb{C} . On the other hand, $\mathfrak{cl}(W, \sigma, F)$ is a right \mathcal{K} -module, in a natural manner. Let \mathcal{S} be a left $\mathfrak{cl}(W, \sigma, F)$ -module, given by

$$\mathcal{S} = \mathfrak{cl}(W, \sigma, F) \otimes_{\mathcal{K}} \mathbb{C}.$$

Definition 3. The constructed module is called *the spinor module* for $\mathfrak{cl}(W, \sigma, F)$ associated to $\{W_1, W_2\}$.

According to Lemma 7, the space \mathcal{S} is naturally identifiable with the exterior algebra W_1^\wedge . In terms of this identification, we have

$$x\xi = x_1 \wedge \xi + x_2 \sqcup \xi,$$

for each $x \in W$, where $x = x_1 + x_2$ and $x_i \in W_i$. In other words, a complete analogy with the classical Cartan formalism holds. The formula (17) corresponds to the classical Cartan map.

Theorem 8. *The algebra $\mathfrak{cl}(W, \sigma, F)$ acts on \mathcal{S} faithfully and irreducibly.*

Proof. We first prove that each vector $\psi \in \mathcal{S} \setminus \{0\}$ is cyclic (which implies that the module is simple). Obviously, the unit element $1_{\mathcal{S}} \in \mathcal{S}$ is cyclic, by construction. The duality between spaces W_1 and W_2 extends to the duality between exterior algebras W_1^\wedge and W_2^\wedge , as explained in Section 3. In terms of this duality, the contraction between elements of the same degree is just the pairing map. It follows that there exists $\varphi \in \mathcal{K}$ such that $\varphi\psi = 1_{\mathcal{S}}$. Hence, ψ is cyclic.

Let us consider an element $x \in \mathfrak{cl}(W, \sigma, F) \setminus \{0\}$. We have

$$\mu^{-1}(x) = \sum_k u_k \otimes v_k + \psi$$

where $\sum_k u_k \otimes v_k \neq 0$ is the component consisting of summands having the minimal second degree n . We can assume that v_k are linearly independent vectors.

There exist spinors $\xi_j \in \mathcal{S}$ satisfying $\psi\xi_j = 0$ and $v_k\xi_j = \delta_{kj}$. This gives $x\xi_j = u_j$ which implies that the representation is faithful. \square

The module \mathcal{S} is completely characterized by the existence of a cyclic vector (the unit element $1_{\mathcal{S}}$), killed by the space W_2 .

In other words let \mathcal{V} be an arbitrary (left) $\mathfrak{cl}(W, \sigma, F)$ -module, possessing a vector v satisfying $\{W_2\}v = \{0\}$. Then there exists the unique module map $\varrho: \mathcal{S} \rightarrow \mathcal{V}$ satisfying $\varrho(1_{\mathcal{S}}) = v$. The map ϱ is injective (because of the simplicity of \mathcal{S}). In particular, if v is cyclic then ϱ is a module isomorphism.

6. CONCLUDING REMARKS

If the braid operator σ is such that $\ker(A)$ is quadratic, then the ideal J_F is generated by elements of the form

$$(18) \quad Q = \psi - F(\psi)1 \otimes 1$$

where $\psi \in W^{\otimes 2}$ is σ -invariant. This covers Clifford algebras based on Hecke braidings [4, 5] (including classical Weyl algebras, in the trivial way).

Quantum Clifford algebras and spinors introduced and analyzed in [1] can be included in the theory presented here. Clifford algebras of [1] are based on Hecke braidings $\tau: V \otimes V \rightarrow V \otimes V$ (where V is a finite-dimensional vector space) admitting extensions to all possible braidings between V and V^* , so that the contraction map is functorial, in the standard sense. Then $W = V \oplus V^*$, the corresponding scalar product F and the braiding σ are expressible in terms of the extended braiding τ and the contraction map.

In the classical theory spinors can be equivalently viewed as elements of the left $\mathfrak{cl}(W, \sigma, F)$ -ideal, generated by a volume element of W_2 . A similar description

is possible in the braided context, if the external algebra W_2^\wedge admits “volume elements”. Namely let us assume that $\omega \in W_2^\wedge$ is such that

$$\{W_2\}\omega = \{0\}.$$

Then the left $\mathfrak{cl}(W, \sigma, F)$ -ideal \mathcal{I}_ω generated by ω is canonically isomorphic to \mathcal{S} , as a left $\mathfrak{cl}(W, \sigma, F)$ -module.

The construction of the map λ_F works for an arbitrary F (and in particular, it is independent of functoriality-type assumptions (10)). For a possibility to define braided Clifford algebras as deformations of braided exterior algebras, it is sufficient to assume that $\ker(A)$ is also a right-ideal in (W^\otimes, \circ) . This assumption is weaker than (10). However, if (10) does not hold, then the symmetry between left and right is broken.

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